Bisemivalues, binomial bisemivalues and multilinear extension for bicooperative games*

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Abstract

We introduce bisemivalues for bicooperative games and we also provide an interesting characterization of this kind of values by means of weighting coefficients in a similar way than given for semivalues in the context of cooperative games. Moreover, the notion of induced bisemivalues on lower cardinalities also makes sense and an adaptation of Dragan's recurrence formula is obtained. Besides its characterization, a computational procedure in terms of the multilinear extension of the game is given

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1 Introduction

The value theory started in 1953 with Shapley [28], who introduced the axiomatic method in game theory to define a solution concept called now the *Shapley value*. The axioms for this value are efficiency,[1] the null player property, symmetry, and additivity. In 1954, Shapley and Shubik [29] applied for the first time the Shapley value as a *power index* (i.e., on simple games). In 1965, Banzhaf [2] defined a different

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[1] This property states that $\sum_{i\in N}\varphi_i[v] = v(N)$ for all $v \in \mathcal{G}_N$.

power index, also obtained in 1971 by Coleman [15] when focussing, rather, on "collective power". In 1975, Owen [25] extended the nonnormalized Banzhaf–Coleman power index to all cooperative games, giving rise to the dummy–independent *Banzhaf value*, which fails to be efficient.

In 1981, Dubey *et al* [17] set aside efficiency and proposed the family of *semivalues*, each one of which is defined by *weighting coefficients* that apply to the marginal contributions $v(S \cup \{i\}) - v(S)$ and are common to all coalitions of a given size. The Shapley value is the only efficient semivalue and the Banzhaf value is the only semivalue with constant coefficients. In 2000, Puente [27] (see also Giménez [20] or Amer and Giménez [1]) defined a special family, *binomial semivalues*: for each one of them, the weighting coefficients depend on a unique parameter $p \in [0,1]$–the Banzhaf value corresponds to $p = 1/2$. A full analysis of these values appears in Carreras and Puente [14]. In 2003, Carreras, Freixas and Puente [13] established the foundations of semivalues as power indices.

In 1988, Weber [30] went further, dropped anonymity, and defined the family of *probabilistic values*, each one of which requires weighting coefficients p_S^i for each player i and each coalition $S \subseteq N \setminus \{i\}$ (of course, anonymity characterizes semivalues within this new family). The payoff that a probabilistic value allocates to each player is thus, again, a weighted sum of his marginal contributions in the game. We quote from Weber [30].

Bicooperative games were introduced by Bilbao *et al* [3] as a generalization of classical cooperative games, where each player can participate positively to the game, negatively, or do not participate. Then, in this kind of games consider pairs (S,T) with $S,\, T \subseteq N$ and $S \cap T = \emptyset$. Thus, (S,T) yields a partition of the set of players N in three groups: (*i*) players in S are defenders of modifying the actual situation and they want to accept a proposal; (*ii*) players in T do not agree with it and they will take actions against any change; and (*iii*) the members of $N \setminus (S \cup T)$ are not convinced of the profits of the change, but they do not stop the actions managed by players in S, that is, they do not have intention of objecting to it. One may think that bicooperative games can be seen as a particular case of games with n players and r alternatives (for $r = 3$), introduced by Bolger in [9] and [10].

A central question in game theory is to define a solution concept for a game, that is, a function which assigns to every game a set of real–valued vectors, each one of them represents a payoff distribution among the players. In the context of bicooperative games this concept has also been studied and different solution concepts have been introduced. In 2000, Bilbao *et al* [4] introduced the Shapley value for bicooperative games. In [5] and [8] Bilbao *et al* introduced the *core*, the *Weber set* and the *selectope* for bicooperative games. In [6] Bilbao *et al* defined and characterized *biprobabilistic values* for bicooperative games following Weber's characterization [30] of

probabilistic values on cooperative games. In 2010 Bilbao *et al* [7] analized *ternary bicooperative games*, which are a refinement of the *ternary voting games* introduced in [18], and defined and axiomatized the Banzhaf power index for these games. Other different definitions of values for bicooperative games can be found in Grabisch and Labreuche [22] and [23]. Borkotokey *et al* [12] prove the existence of a unique potential for bicooperative games and the marginal contribution vector of this potential function coincides with the Shapley value introduced by Bilbao *et al* [4]. In 2012, Borkotokey and Sarmah [11] introduce the notion of a bicooperative game with fuzzy bicoalitions and an explicit form of the Shapley value as a possible solution concept to a particular class of such games is also obtained.
The aim of this paper is to introduce and to characterize bisemivalues for bicooperative games– as a particular family of biprobabilistic values– that parallels the existing statements for semivalues on cooperative games given by Dubey *et al* in [17].
The organization of the paper is as follows. In Section 2, we include a minimum of preliminaries that refers to semivalues for cooperative games and biprobabilistics values for bicooperative games. Section 3 is devoted to define bisemivalues for bicooperative games and to give the main theorem of the paper, that clearly reminds the characterization obtained by Dubey, Neyman and Weber [17] for semivalues on cooperative games. Moreover we deals with *induced bisemivalues* on lower cardinalities and an adaptation of Dragan's recurrence formula [16] is obtained. In Section 4 we introduce the binomial bisemivalues and prove that they are characterized by the (simplest form of) monotonicity of the weighting coefficients, which lie therefore in geometric progression. We also give a computational procedure in terms of the MLE of the game to calculate them. Finally, Section 6 contains two applications of the bisemivalues to the analysis of two examples.

2 Preliminaries

2.1 Cooperative games and semivalues

Let N be a finite set of *players* and 2^N be the set of its *coalitions* (subsets of N). A *cooperative game* on N is a function $v : 2^N \to \mathbb{R}$, that assigns a real number $v(S)$ to each coalition $S \subseteq N$, with $v(\emptyset) = 0$. A game v is *monotonic* if $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$ and *simple* if, moreover, $v(S) = 0$ or 1 for every $S \subseteq N$.
By a *value* on $\mathcal{G}_N$ we will mean a map $f : \mathcal{G}_N \to \mathbb{R}^N$, that assigns to every game v a vector $f[v]$ with components $f_i[v]$ for all $i \in N$.
According Weber's [30] axiomatic description, $\Psi : \mathcal{G}_N \to \mathbb{R}^N$ is a *semivalue* iff it satisfies the following properties:
(i) *linearity*: $\Psi[v+v'] = \Psi[v] + \Psi[v']$ (*additivity*) and $\Psi[\lambda v] = \lambda\Psi[v]$ for all $v, v' \in \mathcal{G}_N$

and $\lambda \in \mathbb{R}$;
(ii) *anonymity*: $\Psi_{\theta i}[\theta v] = \Psi_i[v]$ for all permutation θ on N, $i \in N$, and $v \in \mathcal{G}_N$;
(iii) *positivity*: if v is monotonic, then $\Psi[v] \geq 0$;
(iv) *dummy player property*: if $i \in N$ is a dummy in game v, then $\Psi_i[v] = v(\{i\})$.

There is an interesting characterization of semivalues, by means of *weighting coefficients*, due to Dubey, Neyman and Weber [17]. Set $n = |N|$. Then: (a) for every *weighting vector* $\{p_k\}_{k=0}^{n-1}$ such that $\sum_{k=0}^{n-1} p_k \binom{n-1}{k} = 1$ and $p_k \geq 0$ for all k, the expression

$$\Psi_i[v] = \sum_{S \subseteq N \setminus \{i\}} p_s[v(S \cup \{i\}) - v(S)] \quad \text{for all } i \in N \text{ and all } v \in \mathcal{G}_N,$$

where $s = |S|$, defines a semivalue Ψ; (b) conversely, every semivalue can be obtained in this way; (c) the correspondence given by $\{p_k\}_{k=0}^{n-1} \mapsto \psi$ is bijective.

Thus, the payoff that a semivalue allocates to every player in any game is a weighted sum of his marginal contributions in the game. If p_k is interpreted as the probability that a given player i joins a coalition of size k, provided that all the coalitions of a common size have the same probability of being joined, then $\Psi_i[v]$ is the expected marginal contribution of that player to a random coalition he joins.

Well known examples of semivalues are the *Shapley value* φ (Shapley [28]), for which $p_k = 1/n\binom{n-1}{k}$, and the *Banzhaf value* β (Owen [25]), for which $p_k = 2^{1-n}$. The Shapley value φ is the only *efficient* semivalue, in the sense that $\sum_{i \in N} \varphi_i[v] = v(N)$ for every $v \in \mathcal{G}_N$.

Notice that these values are defined for each N. The same happens with the *binomial semivalues*, introduced by Puente [27] (see also Giménez [20] or Amer and Giménez [1]) as follows. Let $p \in [0,1]$ and $p_k = p^k(1-p)^{n-k-1}$ for $k = 0, 1, \ldots, n-1$. Then $\{p_k\}_{k=0}^{n-1}$ is a weighting vector and defines a semivalue that will be denoted as Ψ^p and called the *p–binomial semivalue*. Using the convention that $0^0 = 1$, the definition makes sense also for $p = 0$ and $p = 1$, where we respectively get the *dictatorial index* Ψ^0 and the *marginal index* Ψ^1, introduced by Owen [26] and such that $\Psi_i^0[v] = v(\{i\})$ and $\Psi_i^1[v] = v(N) - v(N \setminus \{i\})$ for all $i \in N$ and all $v \in \mathcal{G}_N$. Of course, $p = 1/2$ gives $\Psi^{1/2} = \beta$ —the Banzhaf value.

Finally, the *multilinear extension* [2] of a game $v \in \mathcal{G}_N$, introduced by Owen [24], is the real–valued function defined in $\mathbb{R}^n$ by

$$f(x_1, x_2, \ldots, x_n) = \sum_{S \subseteq N} \prod_{i \in S} x_i \prod_{j \in N \setminus S} (1 - x_j) v(S).$$

[2] The term "multilinear" means that, for each $i \in N$, the function is linear in x_i, that is, of the form $f_v(x_1, x_2, \ldots, x_n) = g_i(x_1, x_2, \ldots, \overset{\wedge}{x_i}, \ldots, x_n) x_i + h_i(x_1, x_2, \ldots, \overset{\wedge}{x_i}, \ldots, x_n)$.

As is well known, both the Shapley and Banzhaf values of any cooperative game v can be obtained from its multilinear extension. Indeed, $\varphi[v]$ can be calculated by integrating the partial derivatives of the multilinear extension of the game along the main diagonal $x_1 = x_2 = \cdots = x_n$ of the cube $[0,1]^n$ [24], while the partial derivatives of that multilinear extension, evaluated at point $(1/2, 1/2, \ldots, 1/2)$, give $\beta[v]$ [25]. This latter procedure extends well to any p–binomial semivalue (see Puente [27], Freixas and Puente [19] or Amer and Giménez [1]) by evaluating the derivatives at point $(p, p, \ldots, p)$.

2.2 Bicooperative games and biprobabilistic values

Let N be a finite set of *players* and $3^N = \{(S,T) : S,T \subseteq N, S \cap T = \emptyset\}$ be the set of all ordered pairs of disjoint coalitions. Grabisch and Labreuche [21] proposed a relation in 3^N given by

$$(A,B) \sqsubseteq (C,D) \Leftrightarrow A \subseteq C, B \supseteq D.$$

Following [3], a *bicooperative game* on N is a function $b : 3^N \to \mathbb{R}$, that assigns a real number $b(S,T)$ to each pair of coalitions $(S,T) \in 3^N$, with $b(\emptyset,\emptyset) = 0$. For each $(S,T) \in 3^N$, the worth $b(S,T)$ represents the maximal gain (if $b(S,T) > 0$) or the minimal loss (if $b(S,T) < 0$) that is obtained when players in S are in favor of a change in the situation, players in T are against the change and players in $N \setminus (S \cup T)$ are indifferent. Then $b(\emptyset,N)$ is the cost obtained when all players are against the change and $b(N,\emptyset)$ is the maximal gain obtained when all players want to change the initial situation. We will denote by $\mathcal{BG}_N$ the set of all bicooperative games on N.
A bicooperative game is *monotonic* if $b(S,T) \le b(S',T')$ whenever $(S,T) \sqsubseteq (S',T')$ and *ternary* if, moreover $b(S,T) \in \{-1,0,1\}$ for all $(S,T) \in 3^N$. A *biweighted ternary bicooperative game*, represented by the scheme $b \equiv [[k_1; w_1, ..., w_n], [k_2; m_1, ...m_n]]$, is the ternary bicooperative game defined by

$$b(S,T) = \begin{cases} 1 & \text{if } w(S) \ge k_1 \text{ and } m(T) < k_2 \\ -1 & \text{if } w(S) < k_1 \text{ and } m(T) \ge k_2 \\ 0 & \text{otherwise,} \end{cases}$$

where $w_i > 0$ is the number of votes of player i to approve a decision and $m_i > 0$ is the number of votes of player i to block it and $0 < k_1 \le w(N), 0 < k_2 \le m(N)$,
A player $i \in N$ is a *dummy* in b if $b(S \cup \{i\}, T) = b(S,T) + b(\{i\},\emptyset)$ and $b(S, T \cup \{i\} = b(S,T) + b(\emptyset,\{i\})$ for all $(S,T) \in 3^{N \setminus \{i\}}$, and *null* in b if, moreover, $b(\{i\},\emptyset) = b(\emptyset,\{i\}) = 0$. Two players $i, j \in N$ are *symmetric* in b if $b(S \cup \{i\}, T) = b(S \cup \{j\}, T)$ for all $S \subseteq N \setminus \{i,j\}$ and $b(S, T \cup \{i\}) = b(S, T \cup \{j\})$ for all $T \subseteq N \setminus \{i,j\}$.

Given a nonempty coalition $R \subseteq N$, the restriction to R of a given game b on N is the game $b_{|R}$ on R that we will call a *subgame* of b and is defined by $b_{|R}(S,T) = b(S,T)$ for all $S,T \subseteq R$.
Endowed with the natural operations for real–valued functions, the set of all bicooperative games on N is a vector space $\mathcal{BG}_N$. For every $(S,T) \in 3^N$ such that $(S,T) \neq (\emptyset,\emptyset)$, the *identity game* $\delta_{(S,T)}$ is defined by $\delta_{(S,T)}(A,B) = 1$ if $(A,B) = (S,T)$ and $\delta_{(S,T)}(A,B) = 0$ otherwise and it is easily checked that the set of all identity games is a basis for $\mathcal{BG}_N$, so that $\dim(\mathcal{BG}_N) = 3^n - 1$ if $n = |N|$.
By a *value* on $\mathcal{BG}_N$ we will mean a map $g : \mathcal{BG}_N \to \mathbb{R}^N$, that assigns to every game b a vector $g[b]$ with components $g_i[b]$ for all $i \in N$.
In [6] Bilbao *et al* defined and characterized *biprobabilistic values* for bicooperative games as follows.

Definition 2.1 A value ϕ for player i on $\mathcal{BG}_N$ is a biprobabilistic value if there exist two collections of real numbers $\{p^i_{(S,T)} : (S,T) \in 3^{N\setminus\{i\}}\}$ and $\{q^i_{(S,T)} : (S,T) \in 3^{N\setminus\{i\}}\}$ satisfying $p^i_{(S,T)} \geq 0$, $q^i_{(S,T)} \geq 0$, $\sum_{(S,T)\in 3^{N\setminus\{i\}}} p^i_{(S,T)} = 1$ and $\sum_{(S,T)\in 3^{N\setminus\{i\}}} q^i_{(S,T)} = 1$ such that,

$$\phi_i[b] = \sum_{(S,T)\in 3^{N\setminus\{i\}}} \left[p^i_{(S,T)}(b(S\cup\{i\},T) - b(S,T)) + q^i_{(S,T)}(b(S,T) - b(S,T\cup\{i\})) \right]$$

for every game $b \in \mathcal{BG}_N$.

Notice that $\phi_i[b]$ is a weighted sum of his marginal contributions $b(S\cup\{i\},T) - b(S,T)$, whenever i joins coalition $S \subseteq N \setminus \{i\}$ and his marginal contributions $b(S,T) - b(S,T\cup\{i\})$ whenever i leaves coalition $T\cup\{i\}$, where $p^i_{(S,T)}$ is the probability that player i joins S and $q^i_{(S,T)}$ is the probability that player i leaves $T \cup i$.
Following Bilbao's axiomatic description [6], a value ϕ on $\mathcal{BG}_N$ is a probabilistic value if and only if it satisfies the following properties:
(i) *linearity*: $\phi[b+b'] = \phi[b] + \phi[b']$ (*additivity*) and $\phi[\lambda b] = \lambda\phi[b]$ for all $b,b' \in \mathcal{GB}_N$ and $\lambda \in \mathbb{R}$;
(ii) *positivity*: if b is monotonic, then $\phi[b] \geq 0$;
(iii) *dummy player property*: if $i \in N$ is a dummy in game b, then $\phi_i[b] = b(\{i\},\emptyset) - b(\emptyset,\{i\})$.

3 Bisemivalues for bicooperative games

In this section we introduce and study bisemivalues for bicooperative games. This includes, besides the axiomatic description, a characterization of them by means of

weighting coefficients that parallels the existent characterization of semivalues given by Dubey, Neyman and Weber [17] in the context of cooperative games.
In a similar way to the cooperative case, for the comparison of roles in a game to be meaningful, the evaluation of a particular position should depend on the structure of the game but not on the labels of the players.
From now on we will denote $S \cup \{i\}$ and $S \setminus \{i\}$ by $S \cup i$ and $S \setminus i$ respectively.
In order to define this family we need to introduce a new axiom.

Definition 3.1 *Anonymity axiom.* $\phi_{\pi i}[\pi b] = \phi_i[b]$ for all permutation π over N, $i \in N$, and $b \in \mathcal{BG}_N$, where $\pi b(\pi S, \pi T) = b(S,T)$ and $\pi S = \{\pi i : i \in S\}$.

Now we are ready to introduce bisemivalues on bicooperative games following Weber's axiomatic description of semivalues on cooperative games.

Definition 3.2 A bisemivalue on $\mathcal{BG}_N$ is a map $\psi : \mathcal{BG}_N \to \mathbb{R}^N$ that satisfies linearity, anonymity, positivity and dummy player property.

As we will see, anonymity characterizes bisemivalues within the family of biprobabilistic values.

Theorem 3.3 *A value ψ on $\mathcal{BG}_N$ is a bisemivalue if and only if there exist two collections of real numbers $p_{s,t}$ and $q_{s,t}$ satisfying:*

$$\begin{aligned} & p_{s,t} \geq 0, q_{s,t} \geq 0, \\ & \sum_{s=0}^{n-1} \binom{n-1}{s} \left[\sum_{t=0}^{n-s-1} \binom{n-s-1}{t} p_{s,t} \right] = 1, \\ & \sum_{t=0}^{n-1} \binom{n-1}{t} \left[\sum_{s=0}^{n-t-1} \binom{n-t-1}{s} q_{s,t} \right] = 1, \end{aligned} \tag{1}$$

such that

$$\psi_i[b] = \sum_{(S,T) \in 3^{N \setminus i}} [p_{s,t}(b(S \cup i, T) - b(S,T)) + q_{s,t}(b(S,T) - b(S, T \cup i))]$$

for all $i \in N$ and all $b \in \mathcal{BG}_N$, where $s = |S|$ and $t = |T|$.

Proof ($\Leftarrow$) Taking into account that bisemivalues are a particular case of biprobabilistic values, linearity, dummy and positivity are proved in [6]. Anonimity follows from the fact that the weighting coefficients only depend of the cardinality of S and T

($\Rightarrow$) Following [6], it suffices to prove that if a biprobabilistic value satisfies the anonymity axiom then $p^i_{(S,T)} = p_{s,t}$ and $q^i_{(S,T)} = q_{s,t}$ for all $(S,T) \in 3^{N\setminus i}$ with $s = |S|$ and $t = |T|$, for all $i \in N$.
Let ϕ be a biprobabilistic value then

$$\phi_i[b] = \sum_{(S,T)\in 3^{N\setminus i}} \left[p^i_{(S,T)}(b(S\cup i,T) - b(S,T)) + q^i_{(S,T)}(b(S,T) - b(S,T\cup i)) \right]$$

for each $i \in N$ and for every game $b \in \mathcal{BG}_N$.

For every $(S,T) \in 3^N$ such that $(S,T) \neq (\emptyset,\emptyset)$, the *identity game* $\delta_{(S,T)}$ is defined by

$$\delta_{(S,T)}(A,B) = \begin{cases} 1 & \text{if } (A,B) = (S,T) \\ 0 & \text{otherwise} \end{cases}$$

Notice that $\phi_i(\delta_{(S\cup i,T)}) = p^i_{(S,T)}$ and $\phi_i(\delta_{(S,T\cup i)}) = -q^i_{(S,T)}$ for all $i \in N$ and $(S,T) \in 3^{N\setminus i}$
Let (S,T) and (S',T') be signed coalitions in $3^{N\setminus i}$ such that $(S,T),(S',T') \neq (\emptyset,\emptyset)$ satisfying that $|S| = |S'| < n-1$ and $|T| = |T'| < n-1$. Consider a permutation π over N that takes $\pi S = S'$ and $\pi T = T'$ while living i fixed. Then $\pi\delta_{(S,T)} = \delta_{(S',T')}$. By the anonymity axiom we have

$$\phi_i(\delta_{(S\cup i,T)}) = \phi_i(\delta_{(S'\cup i,T')}) \text{ and } \phi_i(\delta_{(S,T\cup i)}) = \phi_i(\delta_{(S',T'\cup i)})$$

Then $p^i_{(S,T)} = p^i_{(S',T')}$ and $q^i_{(S,T)} = q^i_{(S',T')}$.
Now, let $i,j \in N$, $i \neq j$ and let $(S,T) \in 3^{N\setminus i,j}$. Let us consider the permutation π over N that interchanges i and j while leaving the remaining players fixed. Since $\pi\delta_{(S,T)} = \delta_{(S,T)}$ we have

$$\phi_i(\delta_{(S\cup i,T)}) = \phi_j(\delta_{(S\cup j,T)}) \text{ and } \phi_i(\delta_{(S,T\cup i)}) = \phi_j(\delta_{(S,T\cup j)})$$

Moreover,

$$\phi_i(\delta_{(N,\emptyset)}) = \phi_j(\delta_{(N,\emptyset)}) \text{ and } \phi_i(\delta_{(\emptyset,N)}) = \phi_j(\delta_{(\emptyset,N)})$$

Hence, for every $(S,T) \in 3^{N\setminus i,j}$ there exists $p_{s,t}$ and $q_{s,t}$ such that $p^i_{(S,T)} = p_{s,t}$ and $q^i_{(S,T)} = q_{s,t}$ for all $i \in N$. $\square$

Remark 3.4 (*a*) The payoff that a bisemivalue allocates to every player in any game is a weighted sum of his marginal contributions $b(S\cup\{i\},T) - b(S,T)$ whenever i joins coalition $S \subseteq N\setminus\{i\}$ and his marginal contributions $b(S,T) - b(S,T\cup\{i\})$ whenever i leaves coalition $T\cup\{i\}$, where p_{st} is the probability that player i joins S

and q_{st} is the probability that player i leaves $T \cup i$, provided that all the coalitions of a common size have the same probability of being joined and lived. Notice that among biprobabilistic values, bisemivalues are characterized by the fact that all coalitions of a given size share common weights with regard to all players.
(b) Among bisemivalues, the Shapley value [4], denoted here by φ, for which

$$p_{s,t} = \frac{(n+s-t)!\,(n+t-s-1)!}{(2n)!}\,2^{n-s-t} \text{ and } q_{s,t} = \frac{(n+t-s)!\,(n+s-t-1)!}{(2n)!}\,2^{n-s-t},$$

was characterized by Bilbao *et al* [4] as the only efficient bisemivalue –in the sense that its total power for every $b \in \mathcal{BG}_N$ is $\sum_{i \in N} \varphi_i[b] = b(N,\emptyset) - b(\emptyset,N)$– satisfying the *structural axiom*.
The Banzhaf value [7] denoted here by β, for which $p_{s,t} = q_{s,t} = \left(\frac{1}{3}\right)^{n-1}$ is the only bisemivalue with constant weighting coefficients, that is, weighting coefficients do not depend on the size of the coalitions S and T.
(c) As it is well known, semivalues for cooperative games are defined on cardinalities rather than on specific player sets: this means that a weighting vector $\{p_k\}_{k=0}^{n-1}$ defines a semivalue Ψ on all N such that $n = |N|$. When necessary, we shall write $\Psi^{(n)}$ for a semivalue on cardinality n and p_k^n for its weighting coefficients. A semivalue $\Psi^{(n)}$ induces semivalues $\Psi^{(t)}$ for all cardinalities $t < n$, recurrently defined by the Pascal triangle (inverse) formula given by Dragan [16]:

$$p_k^t = p_k^{t+1} + p_{k+1}^{t+1} \quad \text{for} \quad 0 \le k < t, \tag{2}$$

A series $\Psi = \{\Psi^{(n)}\}_{n=1}^{\infty}$ of semivalues, one for each cardinality, satisfies Dragan's recurrence formula. and we will say that Ψ is a *multisemivalue*. Particularly, the Shapley, the Banzhaf values and all binomial semivalues are multisemivalues.
As we will see, things are very similar to bisemivalues on bicooperative games.

Following Theorem 3.3, analogously to the cooperative case, bisemivalues are also defined on cardinalities rather than on specific player set: that is, two weighting vectors $p_{s,t}$ and $q_{s,t}$ define a bisemivalue ψ on all N such that $n = |N|$. When necessary, we shall write $\psi^{(n)}$ for a bisemivalue on cardinality n, $p_{s,t}^n$ and $q_{s,t}^n$ for its weighting coefficients.

Proposition 3.5 *Given a bisemivalue $\psi^{(n)}$ on $\mathcal{BG}_N$ with weighting coefficients $p_{s,t}^n$ and $q_{s,t}^n$, the recursively obtained numbers*

$$\begin{aligned} p_{s,t}^{m-1} &= p_{s+1,t}^m + p_{s,t}^m + p_{s,t+1}^m, \\ q_{s,t}^{m-1} &= q_{s+1,t}^m + q_{s,t}^m + q_{s,t+1}^m \end{aligned} \tag{3}$$

for $\quad 0 \leq s,t < m \leq n$, *define a induced bisemivalue* $\psi^{(m)}$ *on the space of bicooperative games with m players.*

Proof Let $\psi^{(n)}$ be a bisemivalue with weighting coefficients $p^n_{s,t}$ and $q^n_{s,t}$.
It suffices to prove that if $\psi^{(n)}$ is a bisemivalue on $\mathcal{BG}_N$ then the induced weighting coefficients $p^{n-1}_{s,t}$ and $q^{n-1}_{s,t}$ obtained from (3) define a bisemivalue $\psi^{(n-1)}$ on bicooperative games with $n-1$ players.
We have to check that the induced weighting coefficients satisfy (1). It is straightforward to verify $p^{n-1}_{s,t} \geq 0$ and $q^{n-1}_{s,t} \geq 0$. The remaining condition for the weighting coefficients $p^{n-1}_{s,t}$ follows from the fact that:

$$\begin{aligned}
&\sum_{s=0}^{n-2} \binom{n-2}{s} \left[\sum_{t=0}^{n-s-2} \binom{n-s-2}{t} p^{n-1}_{s,t} \right] \\
&\quad = \sum_{s=0}^{n-2} \binom{n-2}{s} \left[\sum_{t=0}^{n-s-2} \binom{n-s-2}{t} (p^n_{s+1,t} + p^n_{s,t} + p^n_{s,t+1}) \right] \\
&\quad = \sum_{s=0}^{n-2} \binom{n-2}{s} \left[\sum_{t=0}^{n-s-1} \binom{n-s-1}{t} p^n_{s,t} + \sum_{t=0}^{n-s-2} \binom{n-s-2}{t} p^n_{s+1,t} \right] \\
&\quad = \sum_{s=0}^{n-1} \binom{n-1}{s} \left[\sum_{t=0}^{n-s-1} \binom{n-s-1}{t} p^n_{s,t} \right] = 1.
\end{aligned}$$

And analogously for the weighting coefficients $q^{n-1}_{s,t}$. □

Definition 3.6 A series $\psi = \{\psi^{(n)}\}_{n=1}^{\infty}$ of bisemivalues, one for each cardinality, is a *multibisemivalue* if and only if it satisfies (3).

Proposition 3.7 *The expression of the weighting coefficients of any induced bisemivalue* $\psi^{(m)}$ *in terms of the coefficients of the original bisemivalue* $\psi^{(n)}$, *are*

$$\begin{aligned}
p^m_{s,t} &= \sum_{i=0}^{n-m} \binom{n-m}{i} \sum_{j=0}^{n-m-i} \binom{n-m-i}{j} p^n_{s+i,t+j}, \\
q^m_{s,t} &= \sum_{i=0}^{n-m} \binom{n-m}{i} \sum_{j=0}^{n-m-i} \binom{n-m-i}{j} p^n_{s+i,t+j}
\end{aligned} \tag{4}$$

for $\quad 0 \leq s,t < m < n$.

Proof It follows by applying (3) repeatedly. □

4 Binomial bisemivalues

In 2000, Puente [27] (see also Giménez [20] or Amer and Giménez [1]) defined a special family of semivalues on cooperative games, *binomial semivalues*: for each one of them, the weighting coefficients depend on a unique parameter $p \in [0,1]$—the Banzhaf value corresponds to $p = 1/2$. These semivalues are especially suited for the study of cooperative games where the players show some (common) tendency to form coalitions. This tendency is defined by *parameter p*.
Which is the reason for letting p range from 0 to 1? Notice that a reasonable regularity assumption on players' behavior is that the probability to form coalitions follows a monotonic (increasing or decreasing) behavior. Then, the only semivalues such that $p_{k+1} = \lambda p_k$ for all k are precisely the p–binomial semivalues, in which case $\lambda = p/(1-p)$ for each $p \in [0,1]$.
Following this idea, we introduce in this section a subfamily of bisemivalues, called *binomial bisemivalues*. As we will see, they "extend" the concept of binomial semivalues to bicooperative games. The fact that a parameter $p \in [0,1]$ defines this new family of bisemivalues introduces new interesting features in the evaluation of bicooperative games.

Proposition 4.1 *Let $p \in [0,1]$, then the coefficients $p_{s,t} = \frac{p^s(1-p)^{n-s-1}}{2^{n-s-1}}$ and $q_{s,t} = \frac{p^t(1-p)^{n-t-1}}{2^{n-t-1}}$ where $s = |S|$ and $t = |T|$ defines a bisemivalue for the bicooperative games.*

Proof We have to prove that the weighting coefficients satisfy (1). It is straightforward to verify that $p_{s,t} \geq 0$ and $q_{s,t} \geq 0$. The remaining condition for the weighting coefficients $p_{s,t}$ follows from the fact that:

$$\sum_{s=0}^{n-1} \binom{n-1}{s} \left[\sum_{t=0}^{n-s-1} \binom{n-s-1}{t} p_{s,t} \right] = \sum_{s=0}^{n-1} \binom{n-1}{s} \left[\sum_{t=0}^{n-s-1} \binom{n-s-1}{t} \frac{p^s(1-p)^{n-s-1}}{2^{n-s-1}} \right]$$

$$= \sum_{s=0}^{n-1} \binom{n-1}{s} \frac{p^s(1-p)^{n-s-1}}{2^{n-s-1}} \left[\sum_{t=0}^{n-s-1} \binom{n-s-1}{t} \right] = \sum_{s=0}^{n-1} \binom{n-1}{s} p^s(1-p)^{n-s-1} = 1$$

The case of $q_{s,t}$ follows similarly. $\square$

Definition 4.2 *(i)* The dictatorial index D for player $i \in N$ is given by

$$D_i[b] = \frac{1}{2^{n-1}} \sum_{(S,T) \in 3^{N \setminus i}} \left[(b(i,T) - b(\emptyset,T)) + (b(S,\emptyset) - b(S,i)) \right], \text{ for all } b \in \mathcal{BG}_N.$$

(ii) The marginal index M for player $i \in N$ is given by

$$M_i[b] = \sum_{(S,T)\in 3^{N\setminus i}} \left[(b(N,T) - b(N\setminus i,T)) + (b(S,N\setminus i) - b(S,N))\right], \text{ for all } b \in \mathcal{BG}_N.$$

Definition 4.3 Let $p \in [0,1]$. The p–binomial bisemivalue ψ^p on $\mathcal{BG}_N$ is defined by the coefficients $p_{s,t} = \frac{p^s(1-p)^{n-s-1}}{2^{n-s-1}}$ and $q_{s,t} = \frac{p^t(1-p)^{n-t-1}}{2^{n-t-1}}$ where $s = |S|$ and $t = |T|$. In case of $p = 0$ and $p = 1$ using the convention $0^0 = 1$, we respectively obtain the dictatorial index $D = \psi^0$ and the marginal index $M = \psi^1$.

In next proposition we characterize the binomial bisemivalues as the only bisemivalues whose weighting coefficients are in geometric progression, that is, technically they are characterized by the (simplest form of) monotonicity of the weighting coefficients.

Proposition 4.4 *Let $p \in (0,1)$. A bisemivalue ψ on $\mathcal{BG}_N$ is a p-binomial bisemivalue if and only if its weighting coefficients are in geometric progression.*

Proof ($\Rightarrow$) Let ψ^p be a p-binomial bisemivalue, following Definition 4.3, we obtain

$$\frac{p_{s+1,t}}{p_{s,t}} = \frac{q_{s,t+1}}{q_{s,t}} = \frac{2p}{1-p}.$$

($\Leftarrow$) Conversely, if $p_{s+1,t} = kp_{s,t}$ for $0 \le s < n$ and $k > 0$, we can write $p_{s,t} = k^s p_{0,t} = k^s p_{0,0}$ so that the second condition in (1) allows us to determine the parametric expression for the weighting coefficients:

$$\begin{aligned}
1 &= \sum_{s=0}^{n-1} \binom{n-1}{s} \left[\sum_{t=0}^{n-s-1} \binom{n-s-1}{t} p_{s,t}\right] \\
&= \sum_{s=0}^{n-1} \binom{n-1}{s} k^s p_{0,0} \left[\sum_{t=0}^{n-s-1} \binom{n-s-1}{t}\right] \\
&= p_{0,0} \sum_{s=0}^{n-1} \binom{n-1}{s} k^s 2^{n-s-1} = p_{0,0}(2+k)^{n-1}.
\end{aligned}$$

and hence, $p_{0,0} = 1/(2+k)^{n-1} = (1-p)^{n-1}/2^{n-1}$. From the recursive formula $p_{s,t} = k^s p_{0,t} = k^s p_{0,0}$ we obtain the remaining weighting coefficients until $p_{n-1,0} = k^{n-1}/(2+k)^{n-1} = p^{n-1}$.
So, $p = k/(2+k)$. The case of $q_{s,t}$ follows similarly. $\square$

Remark 4.5 The only p–binomial bisemivalues satisfying $p_{s,t} = q_{s,t}$, that is, which weights in a same way the marginal contributions of players in favor or against the change are for $p = 0, 1/3, 1$, corresponding to the dictatorial index, the Banzhaf bisemivalue and the marginal index respectively.

4.1 Computational procedure

The MLE technique has been a useful tool for the calculus of values on cooperative games: it applies to e.g. the Shapley value (Owen [25]), the Banzhaf value (Owen [27]) and all binomial semivalues (Puente [32]). In this section first we introduce the *multilinear extension* of a bicooperative game that parallels the existing *multilinear extension* of a cooperative game given by Owen in [24] and then, we provide a method to compute binomial bisemivalues by means of the multilinear extension of the game.
We identify each $(S,T) \in 3^N$ by vectors (X,Y,Z) of $\mathbb{R}^{3n}$ such that $X = (x_1,\ldots,x_n)$, $Y = (y_1,\ldots,y_n)$, $Z = (z_1,\ldots,z_n)$ and

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}, \quad y_i = \begin{cases} 1 & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad z_i = \begin{cases} 1 & \text{if } i \in N \setminus (S \cup T) \\ 0 & \text{otherwise.} \end{cases}$$

For instead, if $N = \{1,2,3\}$ the coalitions $(\{1,3\},\{2\})$ and $(\{1,2\},\emptyset)$ are identified by $(X,Y,Z) = (1,0,1,0,1,0,0,0,0)$ and $(X,Y,Z) = (1,1,0,0,0,0,0,0,1)$ respectively.
A bicooperative game b is a real-valued function defined on the corners of $[0,1]^{3n}$. We shall extend this function throughout $[0,1]^{3n}$ in a manner that is linear in which variable. This is the *multilinear extension* of b.

Definition 4.6 The *multilinear extension* of a game $b \in \mathcal{BG}_N$ is the real-valued function defined on $\mathbb{R}^{3n}$ by

$$f(X,Y,Z) = \sum_{(S,T)\in 3^N} \left[\prod_{i\in S} x_i \prod_{j\in T} y_j \prod_{k\in N\setminus(S\cup T)} z_k \right] b(S,T). \tag{5}$$

where $X = (x_1,\ldots,x_n)$, $Y = (y_1,\ldots,y_n)$, $Z = (z_1,\ldots,z_n) \in [0,1]^n$.

It is easy to prove that f coincides with b where b is defined.

Proposition 4.7 *If ψ^p is a p–binomial bisemivalue and f is the multilinear extension of a game $b \in \mathcal{BG}_N$ then*

$$\psi_i^{\mathbf{p}}[b] = \left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial z_i}\right)\left(P, \frac{1-P}{2}, \frac{1-P}{2}\right) + \left(\frac{\partial f}{\partial z_i} - \frac{\partial f}{\partial y_i}\right)\left(\frac{1-P}{2}, P, \frac{1-P}{2}\right)$$

for all $i \in N$, where $P = (p, \overset{n}{\ldots}, p)$ and $\dfrac{1-P}{2} = (\frac{1-p}{2}, \overset{n}{\ldots}, \frac{1-p}{2})$.

Proof From Definition 4.6 the partial derivatives of f with respect to x_i, y_i, z_i are:

$$\frac{\partial f}{\partial x_i}(X,Y,Z) = \sum_{(S,T)\in 3^{N\setminus i}} \left[\prod_{j\in S} x_j \prod_{k\in T} y_k \prod_{l\in N\setminus(S\cup T\cup i)} z_l\right] b(S\cup i,T),$$
$$\frac{\partial f}{\partial y_i}(X,Y,Z) = \sum_{(S,T)\in 3^{N\setminus i}} \left[\prod_{j\in S} x_j \prod_{k\in T} y_k \prod_{l\in N\setminus(S\cup T\cup i)} z_l\right] b(S,T\cup i),$$
$$\frac{\partial f}{\partial z_i}(X,Y,Z) = \sum_{(S,T)\in 3^{N\setminus i}} \left[\prod_{j\in S} x_j \prod_{k\in T} y_k \prod_{l\in N\setminus(S\cup T\cup i)} z_l\right] b(S,T),$$

then

$$\frac{\partial f}{\partial x_i}(X,Y,Z) - \frac{\partial f}{\partial z_i}(X,Y,Z) = \sum_{(S,T)\in 3^{N\setminus i}} \left[\prod_{j\in S} x_j \prod_{k\in T} y_k \prod_{l\in N\setminus(S\cup T\cup i)} z_l\right] [b(S\cup i,T) - b(S,T)] \tag{6}$$

$$\frac{\partial f}{\partial z_i}(X,Y,Z) - \frac{\partial f}{\partial y_i}(X,Y,Z) = \sum_{(S,T)\in 3^{N\setminus i}} \left[\prod_{j\in S} x_j \prod_{k\in T} y_k \prod_{l\in N\setminus(S\cup T\cup i)} z_l\right] [b(S,T) - b(S,T\cup i)] \tag{7}$$

Finally, valuating (6) at point $(P, \frac{1-P}{2}, \frac{1-P}{2})$ and (7) at point $(\frac{1-P}{2}, P, \frac{1-P}{2})$ and adding these two results, we obtain the p–binomial bisemivalue

$$\psi_i^p[b] = \sum_{(S,T)\in 3^{N\setminus i}} \left[\frac{p^s(1-p)^{n-s-1}}{2^{n-s-1}}(b(S\cup i,T) - b(S,T)) + \frac{p^t(1-p)^{n-t-1}}{2^{n-t-1}}(b(S,T) - b(S,T\cup i))\right]. \square$$

Corollary 4.8 *If $p = 1/3$, that is $\psi^{1/3} = \beta$ the Banzhaf value, and f is the multilinear extension of a game $b \in \mathcal{BG}_N$ then*

$$\beta_i[b] = \left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial y_i}\right)\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

5 Examples

In this section we present two examples of bicooperative games. The allocations obtained by the players in both cases will be analyzed by using p–binomial bisemivalues and we will compute them by using the MLE technique given in Proposition 4.7.

Example 5.1 (Biweighted ternary bicooperative game) The board of directors of a professional sports club is formed by 11 members. Only the shareholders with a minimum number of shares are represented and the seats are allocated according to the number of shares. At this time there are represented three groups of shareholders: the first with six votes, the second with 4 votes and the third with one vote. At least seven votes are required to approve a proposal and a minimum of 5 votes to block it. This voting system can be modeled by the biweighted ternary bicooperative game

$$b \equiv [[7;6,4,1],[5;6,4,1]]$$

That is, for each $(S,T) \in 3^N$,

$$b(S,T) = \begin{cases} 1 & \text{if } w(S) \geq 7 \text{ and } m(T) < 5 \\ -1 & \text{if } w(S) < 7 \text{ and } m(T) \geq 5 \\ 0 & \text{otherwise.} \end{cases}$$

In this game the worth 1 is attained by the coalitions

$$(\{1,2\},\{3\}),(\{1,3\},\{2\}),(\{1,2\},\emptyset),(\{1,3\},\emptyset) \text{ and } (\{1,2,3\},\emptyset);$$

the worth -1 by coalitions

$$(\emptyset,\{1,2,3\}),(\emptyset,\{1,2\}),(\emptyset,\{1,3\}),(\emptyset,\{2,3\}),(\emptyset,\{1\}),(\{1\},\{2,3\}),$$

$$(\{2\},\{1,3\}),(\{3\},\{1,2\}),(\{2\},\{1\}),(\{3\},\{1\}) \text{ and } (\{2,3\},\{1\});$$

and 0 by the remaining coalitions.

From Definition 4.6 the MLE of b is

$$f(X,Y,Z) = -y_1y_2y_3 - y_1y_2z_3 - y_1y_3z_2 - y_2y_3z_1 - y_1z_2z_3 - x_1y_2y_3 - x_2y_1y_3 - x_3y_1y_2 - x_2y_1z_3 -$$

$$x_3y_1z_2 + x_1x_2z_3 - x_2x_3y_1 + x_1x_3y_2 + x_1x_2z_3 + x_1x_3z_2 + x_1x_2x_3$$

We compute p–binomial bisemivalues by using the MLE technique and Table 1 shows the p–binomial bisemivalues for each player i and for several values of p.

i	$\psi_i^p[v]$	$p = 1/6$	$p = 1/3$ (Banzhaf)	$p = 1/2$
1	$-2p^2+2p+1$	1.2778	1.4444	1.5000
2	$2p(1-p)$	0.2778	0.4444	0.5000
3	$2p(1-p)$	0.2778	0.4444	0.5000

Table 1. p–binomial bisemivalues for several values of p

Example 5.2 Two insurance companies, A_1 and A_2, are always in competition in order to obtain the maximum number of clients in a region. If N is the set of insurance agents, each one of them with an owner clients' list, we can define the bicooperative game $b(S,T)$ as A_1's benefits when players in S work for A_1, players in T work for A_2 and players in $N \setminus (S \cup T)$ do not work for A_1 neither A_2.
Consider $N = \{1,2,3\}$ the number of insurance agents and assume that players 1 and 3 are the agents with the biggest and the smallest clients' list respectively. If an agent lives company A_1, he can go to A_2 and take part or the whole list of his clients or, on the contrary, go to another type of company unrelated to insurances, to be retired, ...
In the first case company A_1 is more damaged than in the second one.
In this situation, let b be the cooperative game defined by

$$\begin{array}{lll}
b(\{1,2,3\},\emptyset) = 100, & b(\emptyset,\emptyset) = 0, & b(\emptyset,\{1,2,3\}) = -60, \\
b(\{1,3\},\emptyset) = 85, & b(\{2,3\},\emptyset) = 75, & b(\{1,2\},\emptyset) = 90, \\
b(\{1,3\},\{2\}) = 50, & b(\{2,3\},\{1\}) = 20, & b(\{1,2\},\{3\}) = 60, \\
b(\{3\},\emptyset) = 65, & b(\{2\},\emptyset) = 70, & b(\{1\},\emptyset) = 80, \\
b(\{3\},\{1\}) = 5, & b(\{3\},\{2\}) = 15, & b(\{2\},\{1\}) = 10, \\
b(\{2\},\{3\}) = 35, & b(\{1\},\{2\}) = 40, & b(\{1\},\{3\}) = 50, \\
b(\{3\},\{1,2\}) = -25, & b(\{2\},\{1,3\}) = -20, & b(\{1\},\{2,3\}) = 5, \\
b(\emptyset,\{1\}) = -30, & b(\emptyset,\{2\}) = -15, & b(\emptyset,\{3\}) = -10, \\
b(\emptyset,\{2,3\}) = -30, & b(\emptyset,\{1,3\}) = -40, & b(\emptyset,\{1,2\}) = -50.
\end{array}$$

Table 2 shows the p–binomial bisemivalues for each player i and for several values of p.

i	$\psi_i^p[v]$	$p = 1/6$	$p = 1/3$ (Banzhaf)	$p = 2/5$	$p = 1/2$
1	$435/4 - 175p/2 + 135p^2/4$	95.1042	83.3333	79.15	73.4375
2	$315/4 - 155p/2 + 135p^2/4$	66.7708	56.6667	53.15	48.4375
3	$65 - 70p + 25p^2$	54.0278	44.4445	41	36.25

Table 2. p–binomial bisemivalues for several values of p

It is easy to check that $\psi_1^p[v] \geq \psi_2^p[v] \geq \psi_3^p[v]$ for all $p \in [0,1]$ and the three players' maximum and minimum allocations, $\psi_i^p[v]$, $i = 1,2,3$, are obtained when $p = 0$ (the dictatorial index) and $p = 1$ (the marginal index), respectively.

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